Fundamentals of Heat and Mass Transfer

Toru Sato

Heat Transfer Equation

- 1^{st} Law of Thermodynamics (Energy Conservation)

$$\frac{D}{Dt}\left[\int_\Omega \rho\left(e+\tfrac{1}{2}u_i^2\right)d\Omega\right]=\int_\Omega \rho g_i u_i d\Omega+\oint_\Gamma \tau_{ij}u_i d\Gamma_j-\oint_\Gamma q_i d\Gamma_i$$

- Differential Form

$$\rho\frac{D}{Dt}\left(e+\tfrac{1}{2}u_i^2\right)=\rho g_i u_i+\frac{\partial}{\partial x_j}\left(\tau_{ij}u_i\right)-\frac{\partial q_i}{\partial x_i}$$

- Kinetic Energy Conservation

$$\rho\frac{D}{Dt}\left(\tfrac{1}{2}u_i^2\right)=\rho g_i u_i+\frac{\partial}{\partial x_j}\left(\tau_{ij}u_i\right)+p\frac{\partial u_i}{\partial x_i}-\mu\Phi$$

$$\Phi=2E_{ij}E_{ij} \qquad E_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial xi}\right)$$

- Navier-Stokes equation (Momentum Conservation)

$$\rho\frac{Du_i}{Dt}=\rho g_i+\frac{\partial \tau_{ij}}{\partial x_j}-\frac{\partial p}{\partial x_i} \qquad \tau_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

- Inner Product with u_i

$$\rho u_i\frac{Du_i}{Dt}=\rho g_i u_i+u_i\frac{\partial \tau_{ij}}{\partial x_j}-u_i\frac{\partial p}{\partial x_i}$$

$$=\rho g_i u_i+\frac{\partial \tau_{ij}u_i}{\partial x_j}-\tau_{ij}\frac{\partial u_i}{\partial x_j}-\frac{\partial p u_i}{\partial x_i}+p\frac{\partial u_i}{\partial x_i}$$

- Kinetic Energy Conservation

$$\rho\frac{D}{Dt}\left(\tfrac{1}{2}u_i^2\right)=\rho g_i u_i+\frac{\partial}{\partial x_j}\left(\tau_{ij}u_i\right)+p\frac{\partial u_i}{\partial x_i}-\mu\Phi$$

- Heat Energy Conservation

$$\rho\frac{De}{Dt}=-\frac{\partial q_i}{\partial x_i}-P\frac{\partial u_i}{\partial x_i}+\mu\Phi$$

- Introducing Enthalpy $\quad h=e+pv \quad P=\rho p$

$$\rho\frac{Dh}{Dt}=\rho\frac{De}{Dt}+\frac{DP}{Dt}+P\frac{\partial u_i}{\partial x_i}=-\frac{\partial q_i}{\partial x_i}+\frac{DP}{Dt}+\mu\Phi$$

- Boussinesq Approximation $\quad h=c_pT \quad \frac{\partial u_i}{\partial x_i}=0$

$$\rho c_p\frac{DT}{Dt}=-\frac{\partial q_i}{\partial x_i}+\mu\Phi$$

- Fourier's Law

$$q_i=-k\frac{\partial T}{\partial x_i}$$

k: thermal conductivity

- Heat Transfer Equation κ : thermal diffusivity

$$\frac{DT}{Dt} = \kappa \frac{\partial^2 T}{\partial x_i^2} \qquad \kappa = \frac{k}{\rho c_p}$$

- Dimensionless Form

$$\frac{DT}{Dt} = \frac{1}{RePr}\frac{\partial^2 T}{\partial x_i^2} = \frac{1}{Pe}\frac{\partial^2 T}{\partial x_i^2}$$

- Prandtl Number, Peclet Number

$$Pr = \frac{\nu}{\kappa} \qquad Pe = \frac{UL}{\kappa}$$

- Heat Convection

$$q_i = \chi(T - T_w) \qquad \chi \text{ : heat transfer coefficient}$$

- Heat Transfer Equation with Heat Convection

$$\frac{DT}{Dt} = -\chi(T - T_w) + \kappa \frac{\partial^2 T}{\partial x_i^2}$$

- Dimensionless Form and Nusselt Number

$$\frac{DT}{Dt} = -\frac{Nu}{RePr}(T - T_w) + \frac{1}{RePr}\frac{\partial^2 T}{\partial x_i^2} \qquad Nu = \frac{L\chi}{\kappa}$$

- Reynolds Averaged Heat Transfer Equation

$$\frac{D\overline{T}}{Dt} = \frac{\partial}{\partial x_i}\left(\kappa \frac{\partial \overline{T}}{\partial x_i} - \overline{u_i' T'}\right)$$

- Eddy Thermal Diffusivity

$$\frac{D\overline{T}}{Dt} = \frac{\partial}{\partial x_i}\left[(\kappa + \kappa_t)\frac{\partial \overline{T}}{\partial x_i}\right]$$

- Dimensionless Form and Turbulent Prandtl Number

$$\frac{D\overline{T}}{Dt} = \frac{\partial}{\partial x_i}\left[\frac{1}{Re}\left(\frac{1}{Pr} + \frac{\nu_t}{Pr_t}\right)\frac{\partial \overline{T}}{\partial x_i}\right] \qquad Pr_t = \frac{\nu_t}{\kappa_t} \approx 1.0$$

Mass Transfer Equation

- Mass Concentration, Mass Fraction

$$\rho = \sum_\alpha \rho^\alpha \left[kg/m^3\right] \qquad Y^\alpha = \frac{\rho^\alpha}{\rho} \qquad \sum_\alpha Y^\alpha = 1$$

- Mass Conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

- Equation of Species Conservation

$$\sum_\alpha \frac{\partial \rho^\alpha}{\partial t} + \sum_\alpha \frac{\partial}{\partial x_i}\left(\rho^\alpha u^\alpha{}_i\right) = \sum_\alpha \dot{m}^\alpha = 0$$

$$\frac{\partial \rho^\alpha}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho^\alpha u^\alpha{}_i\right) = \dot{m}^\alpha$$

- for Mass Fraction

$$v^{\alpha}{}_i = u^{\alpha}{}_i - u_i \quad \text{Diffusive velocity}$$

$$\rho u_i = \sum_{\alpha} \rho^{\alpha} u^{\alpha}{}_i \quad \text{Mass-Average velocity}$$

$$\frac{\partial}{\partial t}\left(\rho Y^{\alpha}\right) + \underline{\frac{\partial}{\partial x_i}\left(\rho Y^{\alpha} u^{\alpha}{}_i\right)} = \dot{m}^{\alpha}$$

$$\underline{\frac{\partial}{\partial t}\left(\rho Y^{\alpha}\right)} + \underline{\frac{\partial}{\partial x_i}\left(\rho Y^{\alpha} u_i\right)} + \frac{\partial}{\partial x_i}\left(\rho Y^{\alpha} v^{\alpha}{}_i\right) = \dot{m}^{\alpha}$$

$$\underline{\rho \frac{\partial Y^{\alpha}}{\partial t}} + Y^{\alpha}\frac{\partial \rho}{\partial t} + \underline{\rho u_i \frac{\partial Y^{\alpha}}{\partial x_i}} + Y^{\alpha}\frac{\partial}{\partial x_i}\left(\rho u_i\right) + \frac{\partial}{\partial x_i}\left(\rho Y^{\alpha} v_i\right) = \dot{m}^{\alpha}$$

$$\rho \frac{DY^{\alpha}}{Dt} + Y^{\alpha}\left(\underline{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho u_i\right)}\right) + \frac{\partial}{\partial x_i}\left(\rho Y^{\alpha} v_i\right) = \dot{m}^{\alpha}$$

$$\rho \frac{DY^{\alpha}}{Dt} + \frac{\partial}{\partial x_i}\left(\rho Y^{\alpha} v_i\right) = \dot{m}^{\alpha}$$

- Fick's Law

$$Y^{\alpha} v^{\alpha}{}_i = -D^{\alpha}\frac{\partial Y^{\alpha}}{\partial x_i} \qquad D^{\alpha}: \text{mass diffusivity}$$

- Mass Transfer Equation

$$\rho \frac{DY^{\alpha}}{Dt} = \frac{\partial}{\partial x_i}\left(\rho D^{\alpha}\frac{\partial Y^{\alpha}}{\partial x_i}\right) + \dot{m}^{\alpha}$$

- If $\rho \approx \text{const}, D^{\alpha} \approx \text{const}$

$$\frac{DC^{\alpha}}{Dt} = D^{\alpha}\frac{\partial^2 C^{\alpha}}{\partial x_i{}^2} + \dot{m}^{\alpha} \qquad C^{\alpha} = \rho^{\alpha}\left[\text{kg}/\text{m}^3\right]$$

- Mass Transfer Equation

$$\frac{DC}{Dt} = D\frac{\partial^2 C}{\partial x_i{}^2}$$

- Dimensionless Form

$$\frac{DC}{Dt} = \frac{1}{ReSc}\frac{\partial^2 C}{\partial x_i{}^2}$$

- Schmitt Number, Lewis Number

$$Sc = \frac{\nu}{D} \qquad Le = \frac{Sc}{Pr} = \frac{\kappa}{D}$$

- Mass Convection

$$m_i = \lambda\left(C - C_w\right) \qquad \lambda: \text{mass transfer coefficient}$$

- Mass Transfer Equation with Mass Convection

$$\frac{DC}{Dt} = -\lambda\left(C - C_w\right) + D\frac{\partial^2 C}{\partial x_i{}^2}$$

- Dimensionless Form and Sherwood Number

$$\frac{DC}{Dt} = -\frac{Sh}{ReSc}\left(C - C_w\right) + \frac{1}{ReSc}\frac{\partial^2 C}{\partial x_i{}^2} \qquad Sh = \frac{L\lambda}{D}$$

- Reynolds Averaged Mass Transfer Equation

$$\frac{D\overline{C}}{Dt} = \frac{\partial}{\partial x_i}\left(D\frac{\partial \overline{C}}{\partial x_i} - \overline{u_i' C'} \right)$$

- Eddy Mass Diffusivity

$$\frac{D\overline{C}}{Dt} = \frac{\partial}{\partial x_i}\left[\left(D + D_t \right)\frac{\partial \overline{C}}{\partial x_i} \right]$$

- Dimensionless Form and Turbulent Schmitt Number

$$\frac{D\overline{C}}{Dt} = \frac{\partial}{\partial x_i}\left[\frac{1}{Re}\left(\frac{1}{SC} + \frac{\nu_t}{Sc_t} \right)\frac{\partial \overline{C}}{\partial x_i} \right] \qquad Sc_t = \frac{\nu_t}{D_t} \approx 1.0$$