Fundamentals of Ocean Model

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Physical Model

Because the density is a function of pressure, temperature, and salinity, the International Equation of State of Sea Water is used.

$$\frac{d\rho}{\rho} = -\alpha dT + \beta dS + \gamma dP$$

where

α : thermal expansion ratio (2.6×10^{-4} 1/°C at 1 atm, 20°C, S=32‰)

β : density change ratio due to salinity (7.6×10^{-4} 1/‰)

γ : compression ratio (inverse of volume expansion ratio) (2.6×10^{-10} 1/Pa)

If the depth is smaller than that of the continental shelf, $\gamma \approx 0$

Original governing equations

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial (u\rho)}{\partial x} + \dfrac{\partial (v\rho)}{\partial y} + \dfrac{\partial (w\rho)}{\partial z} = \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial y} + v\dfrac{\partial \rho}{\partial y} + w\dfrac{\partial \rho}{\partial z} + \rho\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) = 0 \\ \rho\left(\dfrac{\partial (uu)}{\partial x} + \dfrac{\partial (vu)}{\partial y} + \dfrac{\partial (wu)}{\partial z}\right) = -\dfrac{\partial p}{\partial x} + \rho\nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) \\ \rho\left(\dfrac{\partial (uv)}{\partial x} + \dfrac{\partial (vv)}{\partial y} + \dfrac{\partial (wv)}{\partial z}\right) = -\dfrac{\partial p}{\partial y} + \rho\nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}\right) \\ \rho\left(\dfrac{\partial (uw)}{\partial x} + \dfrac{\partial (vw)}{\partial y} + \dfrac{\partial (ww)}{\partial z}\right) = -\dfrac{\partial p}{\partial z} + \rho\nu\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right) - \rho g \end{cases}$$

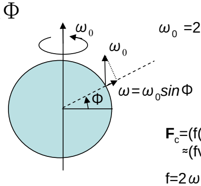
Incompressibility

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\ \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + v\dfrac{\partial \rho}{\partial y} + w\dfrac{\partial \rho}{\partial z} = 0 \\ \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}\right) = -\dfrac{\partial p}{\partial x} + \rho\nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) \\ \rho\left(\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}\right) = -\dfrac{\partial p}{\partial y} + \rho\nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}\right) \\ \rho\left(\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}\right) = -\dfrac{\partial p}{\partial z} + \rho\nu\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right) - \rho g \end{cases}$$

For rotation, the Coriolis force is considered.

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\ \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} - fv\right) = -\dfrac{\partial p}{\partial x} + \rho\nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) \\ \rho\left(\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} + fu\right) = -\dfrac{\partial p}{\partial y} + \rho\nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}\right) \\ \rho\left(\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}\right) = -\dfrac{\partial p}{\partial z} + \rho\nu\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right) - \rho g \end{cases}$$

where $f = 2\omega_0 \sin\Phi$



$\omega_0 = 2\pi$/day$= 0.73 \times 10^{-4}$ (1/s)

$\mathbf{F}_c = (f(v - w\cot\Phi), -f(u - w\cot\Phi), f(\sqrt{(u^2+v^2)}\cot\Phi))$
$\approx (fv, -fu, 0)$

$f = 2\omega$

$\rho \approx \rho_0$

NS equation with Boussinesq approximation
where ρ is constant except for the gravity term

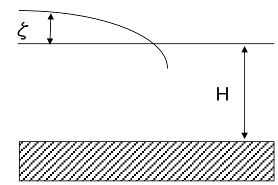
$$\begin{cases} \rho_0\left(\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}-fv\right)=-\dfrac{\partial p}{\partial x}+\rho_0\nu\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right) \\ \rho_0\left(\dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}+fu\right)=-\dfrac{\partial p}{\partial y}+\rho_0\nu\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}+\dfrac{\partial^2 v}{\partial z^2}\right) \\ \rho_0\left(\dfrac{\partial w}{\partial t}+u\dfrac{\partial w}{\partial x}+v\dfrac{\partial w}{\partial y}+w\dfrac{\partial w}{\partial z}\right)=-\dfrac{\partial p}{\partial z}+\rho_0\nu\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}\right)-\rho g \end{cases}$$

Order difference in directions

$L \sim 10^{3\sim6}$m, $H \sim 10^{0\sim3}$m

$w/U \approx H/L$ (due to the Eq. of continuity)→no need to solve w

$$\rho_0\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho_0\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\rho g$$



$$w \ll 1$$
$$0=-\frac{\partial p}{\partial z}-\rho g$$

$$p=-g\int_z^{\zeta}\rho dz=-\rho_0 g(\zeta-z)$$

$$\frac{\partial p}{\partial x}=g\frac{\partial \zeta}{\partial x}$$
$$\frac{\partial p}{\partial y}=g\frac{\partial \zeta}{\partial y}$$

Order difference in directions

$$\nu_H \gg \nu_V$$
$$\nu_H \sim 10^2\ m^2/s$$
$$\nu_V \sim 10^{-2}\ m^2/s$$

Eddy kinematic viscosity

$$\frac{\nu_V}{\nu_{V0}}=(1+5.2Ri)^{-1} \qquad Ri=-\frac{g\dfrac{\partial \rho}{\partial z}}{\rho\left(\dfrac{\partial U}{\partial z}\right)^2}$$

$$\frac{\nu_H}{\nu_{H0}}=L^{-\frac{4}{3}}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$
$$\int_{-H}^{\zeta}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)dz=\int_{-H}^{\zeta}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz+[w]_{-H}^{\zeta}=0$$
$$(H+\zeta)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+w|_{z=\zeta}-w|_{z=-H}=0$$

where u and v are const. in vertical direction (2D approximation)

at bottom
$w|_{z=-H}=0$

at surface
$z-\zeta(x,y,t)=0$
$$\left(\frac{dz}{dt}-\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}\right)|_{z=\zeta}=0$$

$$w|_{z=\zeta}=\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}$$

$$(H+\zeta)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}=0$$
$$\frac{\partial \zeta}{\partial t}+\frac{\partial u(H+\zeta)}{\partial x}+\frac{\partial u(H+\zeta)}{\partial y}=0$$

NS equation with hydrostatic approximation

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} - fv = -\dfrac{g}{\rho_0}\dfrac{\partial \zeta}{\partial x} + \nu_H\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + \nu_V\left(\dfrac{\partial^2 u}{\partial z^2}\right) \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + fu = -\dfrac{g}{\rho_0}\dfrac{\partial \zeta}{\partial y} + \nu_H\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) + \nu_V\left(\dfrac{\partial^2 v}{\partial z^2}\right) \\ \dfrac{\partial \zeta}{\partial t} + \dfrac{\partial u(H+\zeta)}{\partial x} + \dfrac{\partial v(H+\zeta)}{\partial y} = 0 \end{cases}$$

$$\begin{aligned} \nu_H\left(\frac{\partial^2 u}{\partial z^2}\right) &= \frac{1}{HdA}\int\left(\nu_H \frac{\partial^2 u}{\partial z^2}\right) d(HdA) \\ &= \frac{1}{HdA}\int\left(\nu_V \frac{\partial u}{\partial z}\right) n_z dA \\ &= \frac{1}{H}\left(\left(\nu_V \frac{\partial u}{\partial z}\right)_{z=\zeta} - \left(\nu_V \frac{\partial u}{\partial z}\right)_{z=-H}\right) \\ &= \frac{1}{H}\left(C_D \frac{\rho_{air}}{\rho_0} u_{air} U_{air} - \gamma u_B U_B\right) \end{aligned}$$

C_D : surface drag coefficient ~ 0.0015
γ : bottom friction coefficient ~ 0.0025

NS equation with hydrostatic approximation in 2D model

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} - fv = -\dfrac{g}{\rho_0}\dfrac{\partial \zeta}{\partial x} + \nu_H\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + \dfrac{C_D \rho_{air} u_{air} U_{air}}{H\rho_0} - \dfrac{\gamma u_B U_B}{H} \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + fu = -\dfrac{g}{\rho_0}\dfrac{\partial \zeta}{\partial y} + \nu_H\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) + \dfrac{C_D \rho_{air} v_{air} U_{air}}{H\rho_0} - \dfrac{\gamma v_B U_B}{H} \\ \dfrac{\partial \zeta}{\partial t} + \dfrac{\partial u(H+\zeta)}{\partial x} + \dfrac{\partial v(H+\zeta)}{\partial y} = 0 \end{cases}$$

NS equation with hydrostatic approximation in 2.5D model

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} - fv = -\dfrac{g}{\rho_0}\dfrac{\partial \zeta}{\partial x} + \nu_H\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + \nu_V\left(\dfrac{\partial^2 u}{\partial z^2}\right) \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} + fu = -\dfrac{g}{\rho_0}\dfrac{\partial \zeta}{\partial y} + \nu_H\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) + \nu_V\left(\dfrac{\partial^2 v}{\partial z^2}\right) \\ \dfrac{\partial \zeta}{\partial t} + \dfrac{\partial}{\partial x}\int_{-H}^{\zeta} u dz + \dfrac{\partial}{\partial y}\int_{-H}^{\zeta} v dz = 0 \end{cases}$$

Boundary conditions

$$\left(\nu_V \frac{\partial u}{\partial z}\right)_{z=\zeta} = C_D \frac{\rho_{air}}{\rho_0} u_{air} U_{air} \qquad \left(\nu_V \frac{\partial u}{\partial z}\right)_{z=-H} = \gamma u_B U_B$$

$$u_{ns+1} = u_{ns} + dz \frac{C_D}{\nu_V}\frac{\rho_{air}}{\rho_0} u_{air} U_{air} \qquad u_{nb-1} = u_{nb} - dz\frac{\gamma}{\nu_V} u_{nb} U_{nb}$$

Heat and mass transfer equations

$$\frac{d\rho}{\rho} = -\alpha dT + \beta dS + \gamma dP$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0$$

$$\frac{\partial \Phi}{\partial t} + u\frac{\partial \Phi}{\partial x} + v\frac{\partial \Phi}{\partial y} + w\frac{\partial \Phi}{\partial z} = K_H\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) + K_V\left(\frac{\partial^2 \Phi}{\partial z^2}\right)$$

where $\Phi = T, S$

Boundary conditions

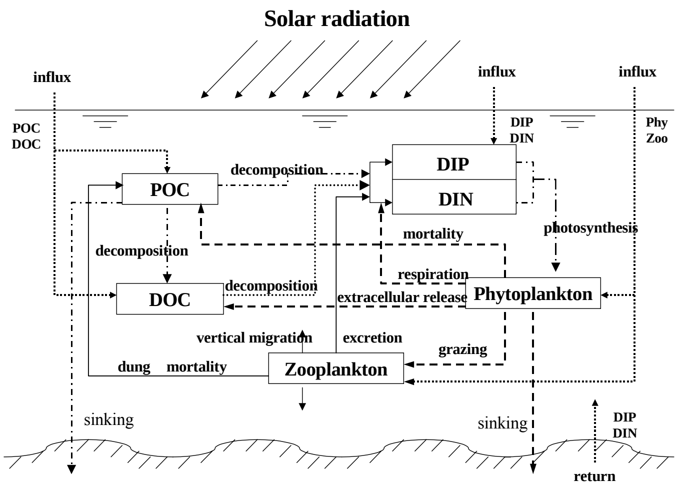
$$Q_T = -Q_{Shortwave} + Q_{longwave} + Q_{latent} + Q_{sensible}$$
$$Q_S = S\,(P_{recipitation} - E_{vapolation})$$

Ecosystem Model

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} - K_B\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) - \frac{\partial}{\partial z}\left(K_B\frac{\partial C}{\partial z}\right) = \left(\frac{dC}{dt}\right)^*$$

Compartments (C)

- phytoplankton (PHY)
- zooplankton (ZOO)
- particulate organic matter (POC)
- dissolved organic matter (DOC)
- dissolved inorganic carbon (DIC)
- dissolved inorganic nitrogen (DIN)
- dissolved inorganic phosphorus (DIP)
- dissolved oxygen (DO)



◆ **DO mg/l**

$$\frac{dDO}{dt} = [TOD:C_P]\cdot(B_1 - B_3) - [TOD:C_Z]\cdot B_9 - [TOD:C_{POM}]\cdot B_{12} - [TOD:C_{DOM}]B_{15} - D_6 + D_7$$

B_1 ： supply by photosynthesis

B_3 ： consumption by respiration of phytoplankton

B_9 ： consumption by respiration of zooplankton

B_{12} ： consumption by making POC inorganic

B_{15} ： consumption by making DOC inorganic

D_6 ： consumption by bottom mud

D_7 ： aeration

Process	Value		
Phytoplankton		Particulate Organic Matter	
Growth Rate	α_1=0.9 β_1=0.063	Decomposition Rate	α_6=0.2 β_6=0.0693
Respiration Rate	α_2=0.03 β_2=0.0519	Sinking Velocity	0.432 m/day
Optimum Light Intensity	200 ly/day	Araction Transfer	ζ =0.25
Extinction Coefficient	k_2=0.1 γ_2=0.0179	$[C_{POM}:P]$	172
(=k_0+ γ [Chl-a:C_p]・P)		$[C_{POM}:N]$	8.4
Day Length)	0.587	Dissolved Organic Matter	
Mortality Rate	α_3=0.05 β_3=0.0693	Decomposition Rate	α_7=0.045 β_7=0.0693
Sinking Velocity	0.173 m/day	$[C_{DOM}:P]$	337
$[C_p:P]$	117	$[C_{DOM}:N]$	11.2
$[C_p:N]$	7.1	Others	
$[TOD:C_p]$	0.00347	Half Saturation Constant	DO_1= DO_2=1 mg/l
$[C_p:Chl\text{-}a]$	50	Reaeration Coefficient	k_a=0.15
Half Saturation Coefficient	K_N :3.0(μg・at/l)	Dissolution rate	Phosphorus：2.45
Half Saturation Coefficient	K_P :0.1(μg・at/l)		Ammonium：24.5
Extracellular Release	13.50%		mg/m^2・day
Zooplankton		Oxygen Consumption	1500 mgO_2/m^2・day
Grazing Rate	α_4=0.28 β_4=0.0693	In Mud	
Ivlev Coefficient	0.082	$[TOD:C_Z]$	0.000347
Mortality Rate	α_5=0.04725 β_5=0.0693	$[TOD:C_{POM}]$	0.000347
Threshold Concentration	50 mgC/m^3	$[TOD:C_{DOM}]$	0.000347
Assimilation Efficiency	0.7		
Total Growth Efficiency	0.3		
$[C_p:P]$	124		
$[C_p:N]$	6.34		

parameters (Nakata, 1993)