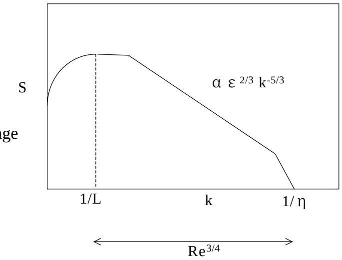
Environmental Fluid Modelling
Turbulence

Toru Sato

(1) Energy Cascade and Kolmogorov's Law

- Energy Cascade

Energy Contain Range
Inertial Subrange
Viscous Dissipation Range



- Energy Contain Range

$$F\left(\frac{k}{k_0}\right)=\frac{\varepsilon(k)}{Kl}$$

- Inertial Subrange

  Kolmogorov's 1st Law (equilibrium, isotropic)

  Kolmogorov's 2nd Law (independence of viscosity)

(2) RANS(Reynolds Averaged NS) Equation

- Reynolds Average

$$U_i=\overline{U}_i+u_i \qquad P=\overline{P}+p$$

- NS Eq for Average

$$\frac{\partial \overline{U}_i}{\partial t}+\overline{U}_j\frac{\partial \overline{U}_i}{\partial x_j}=\frac{\partial}{\partial x_j}\left(-\overline{P}\delta_{ij}+\nu\overline{E}_{ij}-\overline{u_i u_j}\right)$$

- NS Eq for Fluctuation

$$\frac{\partial u_i}{\partial t}+\overline{U}_j\frac{\partial u_i}{\partial x_j}=-u_j\frac{\partial u_i}{\partial x_j}-u_j\frac{\partial \overline{U}_i}{\partial x_j}+\frac{\partial}{\partial x_j}\left(-p\delta_{ij}+\nu e_{ij}+\overline{u_i u_j}\right)$$

- Transport Eq for Reynolds Stress

$$\frac{\partial \overline{u_i u_j}}{\partial t}+\overline{U}_k\frac{\partial \overline{u_i u_j}}{\partial x_k}=\frac{\partial}{\partial x_k}\left(-\overline{pu_j}\delta_{ik}-\overline{pu_i}\delta_{jk}+\nu\frac{\partial \overline{u_i u_j}}{\partial x_k}-\overline{u_i u_j u_k}\right)$$
$$-\overline{u_j u_k}\frac{\partial \overline{U}_i}{\partial x_k}-\overline{u_i u_k}\frac{\partial \overline{U}_j}{\partial x_k}+\overline{pe_{ij}}-2\nu\overline{e_{ik}e_{jk}}$$

- Transport Eq for Turbulence Kinetic Energy

$$\frac{\partial K}{\partial t}+\overline{U}_j\frac{\partial K}{\partial xj}=\frac{\partial}{\partial x_j}\left(-\overline{pu_j}+\nu\frac{\partial K}{\partial xj}-\tfrac{1}{2}\overline{Ku_j}\right)-u_j\frac{\partial K}{\partial x_j}-2\nu\overline{e_{ij}e_{ij}}$$

(3) Eddy Viscosity

- Analogy from Molecular Viscosity

$$R_{ij}=-\overline{u_i u_j}=\nu_t\overline{E}_{ij}-\tfrac{2}{3}K\delta_{ij}$$

- Mixing Length Model

$$\nu_t=ku_t l_m=k\frac{l_m{}^2}{t}=kl_m{}^2\left|\frac{d\overline{U}}{dy}\right|$$
$$\nu_t=ku_t l_m=K^{1/2}l_m$$
$$\nu_t=ku_t l_m=CK^{1/2}\frac{K^{3/2}}{\varepsilon}=\frac{CK^2}{\varepsilon}$$

(4) Turbulence Models for RANS

- 0-Equation Model

  Baldwin-Lomax Model $\nu_t = \min(\nu_{ti}, \nu_{to})$

  (no separation, thin boundary layer)

$$\nu_{ti} = l^2|\omega| \qquad l = ky\left[1-\exp\left(-\frac{y^+}{26}\right)\right]$$

$$\nu_{to} = KC_{cp}F_{wake}F_{kleb}$$

$$F_{wake} = y_{\max}F_{\max} \qquad F_{kleb} = \left[1+5.5\left(\frac{C_{kleb}y}{F_{\max}}\right)\right]$$

$$F = y|\omega|\left[1-\exp\left(-\frac{y^+}{26}\right)\right]$$

- 1-Equation Model $\nu_t = lu_t = lK^{0.5}$
- 2-Equation Model

  K-epsilon Model $\nu_t = C\dfrac{K^2}{\varepsilon}$ $\left(\varepsilon = \nu\overline{\dfrac{\partial u_i}{\partial x_k}\dfrac{\partial u_i}{\partial x_k}}\right)$

$$\frac{\partial K}{\partial t} + \overline{U_k}\frac{\partial K}{\partial x_k} = -u_j u_k\frac{\partial K}{\partial x_k} + \frac{\partial}{\partial x_k}\left(-\overline{pu_k} + \nu\frac{\partial K}{\partial x_k} - \tfrac{1}{2}\overline{Ku_k}\right) - \nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}}$$

$$= \tfrac{1}{2}\nu_t\overline{E_{ij}E_{ij}} + \frac{\partial}{\partial x_k}\left((\nu + \frac{\nu_t}{\sigma})\frac{\partial K}{\partial x_k}\right) - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \overline{U_k}\frac{\partial \varepsilon}{\partial x_k} = \frac{\varepsilon}{K}\left(C_1\tfrac{1}{2}\nu_t\overline{E_{ij}E_{ij}} - C_2\varepsilon\right) + \frac{\partial}{\partial x_k}\left((\nu + \frac{\nu_t}{\sigma})\frac{\partial \varepsilon}{\partial x_k}\right)$$

Low Reynolds Number Modification

(damping effect near wall)

$$\frac{\partial K}{\partial t} + \overline{U_k}\frac{\partial K}{\partial x_k} = \tfrac{1}{2}\nu_t\overline{E_{ij}E_{ij}} + \frac{\partial}{\partial x_k}\left((\nu + \frac{\nu_t}{\sigma})\frac{\partial K}{\partial x_k}\right) - \varepsilon' - D$$

$$\frac{\partial \varepsilon'}{\partial t} + \overline{U_k}\frac{\partial \varepsilon'}{\partial x_k} = \frac{\varepsilon'}{K}\left(C_1 f_1\tfrac{1}{2}\nu_t\overline{E_{ij}E_{ij}} - C_2 f_2\varepsilon\right) + \frac{\partial}{\partial x_k}\left((\nu + \frac{\nu_t}{\sigma})\frac{\partial \varepsilon'}{\partial x_k}\right) + F$$

$$\varepsilon' = \varepsilon - 2\nu\left(\frac{\partial K^{0.5}}{\partial y}\right)^2$$

Two-Layer K-epsilon

(inner: 0-eq model, outer: K-epsilon)

(5) LES (Large Eddy Simulation)

- Spatial Filtering (GS and SGS components)

  Gaussian

$$\hat{G}(k_i) = \exp\left[-\frac{(\Delta_i x_i)^2}{24}\right]$$

  Spectral Cut-Off

$$\hat{G}(k_i) = \begin{cases} 1 & \left(|k_i| \le \dfrac{\pi}{\Delta_i}\right) \\ 0 & \left(|k_i| > \dfrac{\pi}{\Delta_i}\right) \end{cases}$$

  Top-Hat

$$G(x_i) = \begin{cases} \dfrac{1}{\Delta_i} & \left(|x_i| \le \dfrac{\Delta_i}{2}\right) \\ 0 & \left(|x_i| > \dfrac{\Delta_i}{2}\right) \end{cases}$$

- Local Equilibrium $\varepsilon = \tau_{it}\overline{S_{ij}}$
- Eddy Viscosity $\tau_{ij} = 2\nu_t\overline{S_{ij}}$
- Dimension Analysis

$$\nu_t = C_\nu K^{0.5}\Delta \qquad \varepsilon = -C_\varepsilon\frac{K^{1.5}}{\Delta}$$

- Smagorinsky Model

$$\nu_t = (C_s\Delta)^2\sqrt{2\overline{S_{ij}S_{ij}}}$$

$$\varepsilon = -2\nu_t\overline{S_{ij}S_{ij}} = -2\sqrt{2}(C_s\Delta)^2\left(\overline{S_{ij}S_{ij}}\right)^{3/2}$$

- Kolmogorov's –5/3 Law $S(k) = \alpha\varepsilon^{2/3}k^{-5/3}$
- Integration of Energy Spectrum

$$\overline{S_{ij}S_{ij}} = \int_0^{\pi/4} k^2 S(k)dk = \frac{3}{4}\alpha|\varepsilon|^{2/3}\left(\frac{\pi}{\Delta}\right)^{4/3}$$

- Then we get $C_s \approx 0.2$
- Van Driest's Damping Fanction near Wall

$$\Delta' = \Delta\left[1-\exp\left(-\frac{y^+}{26}\right)\right]$$