Fundamentals of Viscous Fluid Flow

Toru Sato

Inviscid and Viscous

- inviscid ⇔ viscous
- ideal / perfect fluid ⇔ real fluid
- Euler equation ⇔ Navier-Stokes equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} \qquad \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

in tensor form

Inviscid and Viscous

- inviscid ⇔ viscous

viscous: sticky, greasy, adhesive,

friction, no-slip,

viscosity: relationship between stress and strain rate

(elasticity: relationship between stress and strain)

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

density (kg/m^3)

viscosity (kg/m/s=Pa s)

$\mu = \rho\nu$

kinetic viscosity (m^2/s)

Which is more viscous, air or water?

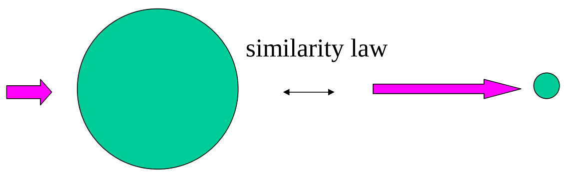
5℃	ν (m^2/s)	ρ (kg/m^3)	μ (Pa s)
air	13.8×10^{-6}	1.27	17.5×10^{-6}
water	1.52×10^{-6}	1000.0	1.52×10^{-3}

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j{}^2}$$

dimensionless form $u = u'U,\ \ x = x'L,\ \ t = t'\frac{L}{U},\ \ p = p'\rho U^2$

$$\frac{\partial u'_i}{\partial t'} + u'_j \frac{\partial u'_i}{\partial x'_j} = -\frac{\partial p'}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u'_i}{\partial x'_j{}^2}$$

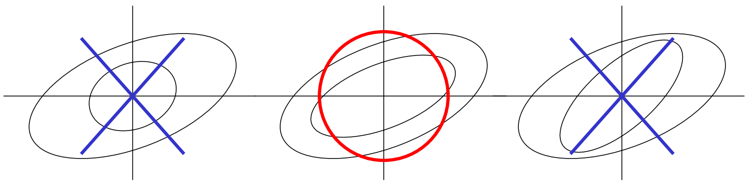
$$\boxed{Re = \frac{\rho U L}{\mu} = \frac{UL}{\nu}}$$ Reynolds number



Newtonian Fluid

- Newton’s Law in viscosity: relationship between stress and strain rate is linear.

$$\tau_{ij} = \nu d_{ij} = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\right) \qquad \because d_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$



Navier-Stokes Equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} \qquad \because \sigma_{ij} \equiv -p\delta_{ij} + \tau_{ij}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \qquad \text{stress-divergence form}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j{}^2} \qquad \because \frac{\partial u_j}{\partial x_j} = 0$$

continuum equation for incompressible fluid

Irrotational Fluid

- “irrotational” means “no vortices”
- deformation of NS-equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{g} \qquad \mathbf{g} \equiv (0,0,-g)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla(\mathbf{uu}) = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{g} \qquad \because \nabla\cdot\mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times(\nabla\times\mathbf{u}) = -\frac{1}{\rho}\nabla p + \nu\nabla\times(\nabla\times\mathbf{u}) + \mathbf{g}$$

rotational form

$$\because \nabla(\mathbf{ab}) = \tfrac{1}{2}[\nabla(\mathbf{a}\cdot\mathbf{b}) - \nabla\times(\mathbf{b}\times\mathbf{a}) - \mathbf{b}\times(\nabla\times\mathbf{a}) - \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\nabla\cdot\mathbf{a} + \mathbf{a}\nabla\cdot\mathbf{b}]$$

$$\because \nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla\cdot\nabla\mathbf{a}$$

$$\frac{\partial \mathbf{u}}{\partial t}+\nabla\left(\frac{1}{2}\mathbf{u}^2\right)-\mathbf{u}\times\boldsymbol{\omega}=-\frac{1}{\rho}\nabla p+\nu\nabla\times\boldsymbol{\omega}+\mathbf{g} \qquad \because \boldsymbol{\omega}\equiv\nabla\times\mathbf{u}$$

$$\frac{\partial \mathbf{u}}{\partial t}+\nabla\left(\frac{1}{2}\mathbf{u}^2+\frac{p}{\rho}-\mathbf{g}\cdot\mathbf{x}\right)=\mathbf{u}\times\boldsymbol{\omega}+\nu\nabla\times\boldsymbol{\omega}$$

if irrotational (at the same time, inviscid)

$$\frac{\partial \mathbf{u}}{\partial t}+\nabla\left(\frac{1}{2}\mathbf{u}^2+\frac{p}{\rho}-\mathbf{g}\cdot\mathbf{x}\right)=0 \qquad \boldsymbol{\omega}=0$$

$$\nabla\left(\frac{\partial \Phi}{\partial t}+\frac{1}{2}\mathbf{u}^2+\frac{p}{\rho}-\mathbf{g}\cdot\mathbf{x}\right)=0 \qquad \because \nabla\Phi=\mathbf{u}$$

Φ : velocity potential

$$\frac{\partial \Phi}{\partial t}+\frac{1}{2}\mathbf{u}^2+\frac{p}{\rho}+gz=\text{const.}$$

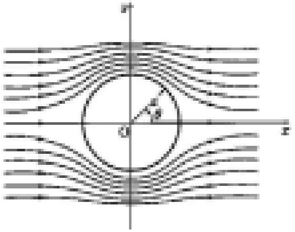
Bernoulli's theorem

$$\nabla^2\Phi=\nabla\cdot\nabla\mathbf{u}=\nabla(\nabla\cdot\mathbf{u})=0$$

Laplace eq. for velocity potential = continuum eq. of incompressible fluid

What is all about Viscous Fluid?

if the flow is irrotational (and inviscid)...



velocity potential for a cylinder in uniflow $\Phi=U\left(z+\frac{a^2}{z}\right)$

in cylindrical coordinates $\Phi=Ur\left(1+\frac{a^2}{r^2}\right)\cos\theta$

solution

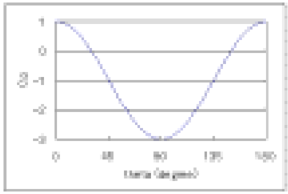
$$v_\theta=\frac{1}{r}\frac{\partial\Phi}{\partial\theta}=-U\left(1+\frac{a^2}{r^2}\right)\sin\theta$$

on the surface of cylinder, where r=a

$$v_\theta=-2U\sin\theta$$

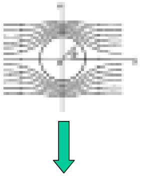
Bernoulli's theorem

$$\begin{aligned}p_\theta&=\tfrac{1}{2}\rho\left(U^2-v_\theta{}^2\right)\\&=\tfrac{1}{2}\rho U^2\left(1-4\sin^2\theta\right)\\&=\tfrac{1}{2}\rho U^2\left(2\cos2\theta-1\right)\end{aligned}$$



No Drag : D'Alembert's paradox

if the flow is viscous ...



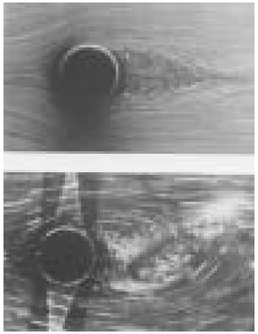
Friction

$$F=\mu\left.\frac{\partial u}{\partial y}\right|_{y=0} \qquad (u=v=0)$$

Separation

$Re=5.5$

$Re=202$



Why and how the flow separates?

NS equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

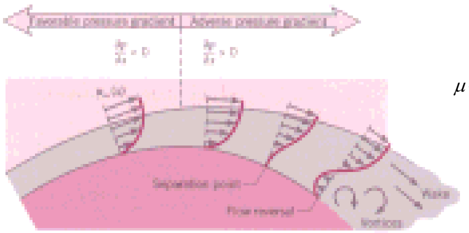
At the solid surface (at y=0)

$(u=v=0)$

$$\frac{\partial p}{\partial x}=\mu\frac{\partial^2 u}{\partial y^2}$$

$\mu\frac{\partial^2 u}{\partial y^2}<0$

$(\frac{\partial p}{\partial x}<0)$

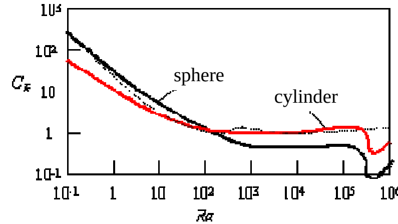


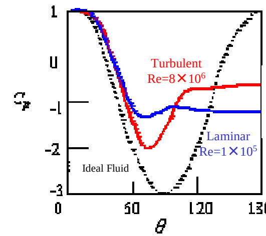
$\mu\frac{\partial^2 u}{\partial y^2}>0$

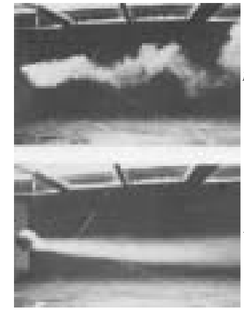
$(\frac{\partial p}{\partial x}>0)$

FIGURE 7.6 Velocity profile associated with separation on a circular cylinder in cross flow.

Pressure drag







$Re=2.4\times10^4$

$Re=6.0\times10^5$