

第7回目の内容

非圧縮粘性流体の運動

- 粘性流体に働く応力
- 粘性流体の流れの支配方程式
- 粘性流体の流れの例

粘性流体に働く応力

理想流体(非粘性流体)では、流体要素に働く応力は圧力による垂直応力だけ。流体要素の変形に伴う応力はない。

粘性流体では、圧力による垂直応力に加えて、「変形に抵抗する力」である粘性応力が働く。粘性応力には検査面に平行な成分「剪断応力」と検査面に垂直な成分「垂直応力」がある。

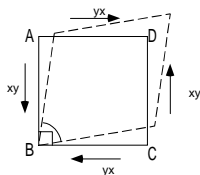
応力の表記

τ_{xy} : x軸に垂直な検査面におけるy軸方向の応力

$\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy}$: 剪断応力

$\tau_{xx}, \tau_{yy}, \tau_{zz}$: 垂直応力

粘性剪断応力: 面に平行に働く応力



剪断変形により生じる

$$\tau_{xy} = \tau_{yx} = -\mu \frac{\partial \gamma}{\partial t} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

μ は粘性係数(流体の物性値)

垂直応力: 面に垂直に働く応力

伸縮変形により生じる。

$\Delta h \cdot \frac{\partial u}{\partial x} \cdot \Delta t$
 $\frac{1}{2} \Delta h \cdot \frac{\partial u}{\partial x} \cdot \Delta t$
 $\Delta h \cdot \frac{\sqrt{2}}{2} \cdot \Delta \alpha = \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \cdot \Delta h \cdot \frac{\partial u}{\partial x} \cdot \Delta t$
 $\frac{\Delta \alpha}{\Delta t} = \frac{1}{2} \frac{\partial u}{\partial x}$
 $\frac{\partial \alpha}{\partial t} = \frac{1}{2} \frac{\partial u}{\partial x} \cdot \frac{\partial \gamma}{\partial t} = -2 \frac{\partial \alpha}{\partial t} \cdot \frac{\partial \gamma}{\partial t} = -\frac{\partial u}{\partial x}$
 $\tau = -\mu \frac{\partial \gamma}{\partial t} = \mu \frac{\partial u}{\partial x}$
 $\frac{\sqrt{2}}{2} \Delta h \tau \frac{1}{\sqrt{2}} = \tau_{xx} \frac{\Delta h}{2} \quad \tau = \tau_{xx} = \mu \frac{\partial u}{\partial x}$ (引張)
 $-\frac{\sqrt{2}}{2} \Delta h \tau \frac{1}{\sqrt{2}} = \tau_{yy} \frac{\Delta h}{2} \quad -\tau = \tau_{yy} = -\mu \frac{\partial u}{\partial x}$ (圧縮)

伸びる方向に引張、他の方向に圧縮の応力となる。

$$\tau_{xx} = \underbrace{\mu \frac{\partial u}{\partial x}}_{x\text{方向の伸び:引張}} - \underbrace{\mu \frac{\partial v}{\partial y}}_{y\text{方向の伸び:圧縮}} - \underbrace{\mu \frac{\partial w}{\partial z}}_{z\text{方向の伸び:圧縮}}$$

$$= 2\mu \frac{\partial u}{\partial x} - \mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

非圧縮性流体では0

$$\left. \begin{aligned} \tau_{xx} &= 2\mu \frac{\partial u}{\partial x} \\ \tau_{yy} &= 2\mu \frac{\partial v}{\partial y} \\ \tau_{zz} &= 2\mu \frac{\partial w}{\partial z} \end{aligned} \right\} \text{粘性による垂直応力}$$

5

応力テンソル

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \mu \begin{pmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} + \frac{\partial w}{\partial z} \end{pmatrix} = \mathbf{T}$$

$$(x, y, z) = (x_1, x_2, x_3) = \mathbf{x}$$

$$(u, v, w) = (u_1, u_2, u_3) = \mathbf{u}$$

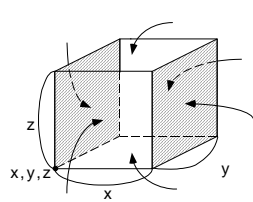
と書くと

$$\tau_{x_i x_j} = \tau_{j i} = \tau_{ji} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

応力テンソル \mathbf{T} と検査面 A に働く応力 \mathbf{T}_A の関係は、
 $\mathbf{T} \cdot \mathbf{n}_A = \mathbf{T}_A$ (\mathbf{n}_A は面 A の単位法線ベクトル)

6

流体要素に働く力 = 密度・加速度の関係から支配方程式を導く



x方向に働く力は

$$\begin{aligned} & \left[\underbrace{-(-p + \tau_{xx})}_{\text{面の垂直応力}} + \underbrace{\left[-\left(p + \frac{\partial p}{\partial x} \Delta x \right) + \tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} \Delta x \right]}_{\text{面の垂直応力}} \right] \Delta y \Delta z \\ & + \left[\underbrace{-\tau_{yx}}_{\text{面のせん断応力}} + \underbrace{\left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \Delta y \right]}_{\text{面のせん断応力}} \right] \Delta x \Delta z \\ & + \left[\underbrace{-\tau_{zx}}_{\text{面のせん断応力}} + \underbrace{\left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \Delta z \right]}_{\text{面のせん断応力}} \right] \Delta x \Delta y \\ & = \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \Delta x \Delta y \Delta z \end{aligned}$$

応力の勾配が単位体積に
 働く力となる。

7

同様にy方向に作用する力 $\left(-\frac{\partial p}{\partial y} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) \Delta x \Delta y \Delta z$

z方向に作用する力 $\left(-\frac{\partial p}{\partial z} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) \Delta x \Delta y \Delta z$

流体単位体積に作用する力

$$\begin{aligned} x\text{方向} : & -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left(2\mu \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) + \frac{\partial}{\partial z} \left(\mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right) \\ & = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\ y\text{方向} : & -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left(\mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) + \frac{\partial}{\partial y} \left(2\mu \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left(\mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \right) \\ & = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\ z\text{方向} : & -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x} \left(\mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right) + \frac{\partial}{\partial y} \left(\mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \right) + \frac{\partial}{\partial z} \left(2\mu \frac{\partial w}{\partial z} \right) \\ & = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \end{aligned}$$

8

非圧縮 オイラー(Euler)方程式 (非粘性)

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + f_x$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + f_y \quad (f_x, f_y, f_z) \text{は単位質量あたりの外力}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + f_z$$

非圧縮 ナビエ・ストークス(Navier-Stokes)方程式 (粘性)

$$\begin{aligned} \rho \frac{Du}{Dt} &= \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + f_x \\ \rho \frac{Dv}{Dt} &= \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + f_y \\ \rho \frac{Dw}{Dt} &= \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + f_z \end{aligned}$$

対流項

圧力項

粘性項

外力項

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

9

連続の式は粘性流体でも非粘性流体でも同じ。

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

2次元のな流れでは、一様な方向にz軸を取ると、

$$w = 0, \quad \frac{\partial}{\partial z} = 0 \text{より}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + f_x$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + f_y$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

11

テンソルで書くと、

$$\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot (\boldsymbol{\tau} - p\mathbf{I}) + \mathbf{f} \quad (\text{密度} \cdot \text{加速度} = \text{応力の発散} + \text{外力})$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\left[\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \quad \boldsymbol{\tau} = \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \quad -p\mathbf{I} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}, \quad \mathbf{f} \text{は外力ベクトル} \right]$$

連続の式を使うと、

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}$$

10

また、円筒座標系では、

非圧縮 連続の式

$$\frac{\partial(v_r \cdot r)}{r \partial r} + \frac{\partial v_\theta}{r \partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

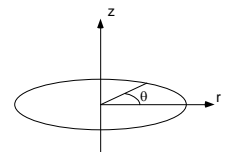
非圧縮 N.S式

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{\partial v_r}{r \partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right) + f_r$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + v_\theta \frac{\partial v_\theta}{r \partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_\theta v_r}{r} = -\frac{1}{\rho} \frac{\partial p}{r \partial \theta} + \nu \left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right) + f_\theta$$

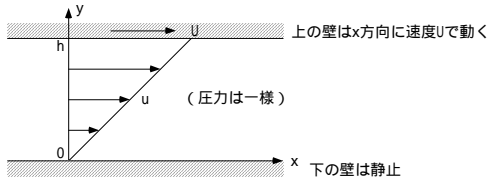
$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_\theta \frac{\partial v_z}{r \partial \theta} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 v_z + f_z$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$



12

粘性流体の運動 (1)クエット流れ

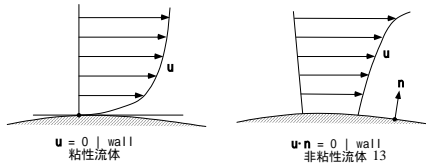


粘性流体は壁面において「すべり」がない。
(壁面では流体粒子は壁と一緒に動く)

境界条件は

$$y = h \text{ で } u = U, v = 0$$

$$y = 0 \text{ で } u = 0, v = 0$$



$u = 0 \mid \text{wall}$
粘性流体

$u \cdot n = 0 \mid \text{wall}$
非粘性流体 13

x方向のN.S式は、

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + f_x$$

外力なし $\rightarrow f_x = 0$

x方向への変化なし $\rightarrow \frac{\partial}{\partial x} = 0, \frac{\partial^2}{\partial x^2} = 0$

時間変化なし(定常) $\rightarrow \frac{\partial}{\partial t} = 0$

y方向への流れなし $\rightarrow v = 0$

よって、

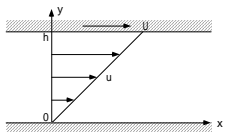
$$\mu \frac{\partial^2 u}{\partial y^2} = 0$$

$$u = C_1 y + C_2$$

境界条件 $u = 0 \mid_{y=0}, u = U \mid_{y=h}$ より、

$$u = \frac{U}{h} y \quad \text{直線速度分布}$$

別の考え方 (N.S式を使わないで解くと)



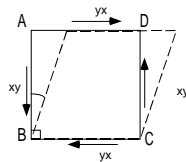
x軸方向に一樣な流れで、 $v=0$ であれば流体要素の伸縮はないので
(正確にはx軸方向及びy軸方向に伸縮はない)粘性応力は、

$$\tau_{yx} = \tau_{xy} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = \mu \frac{\partial u}{\partial y}$$

右の流体要素に働く力の釣り合いより、

$$\frac{\partial \tau_{yx}}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) = 0$$

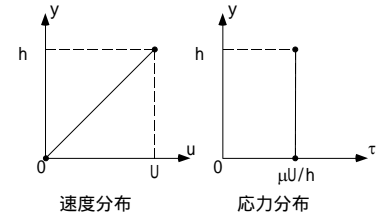
$$\therefore \frac{\partial^2 u}{\partial y^2} = 0$$



$$\tau_{yx} = \tau = \mu \frac{\partial u}{\partial y}$$

$$u = \frac{U}{h} y \text{ なので}$$

$$\tau = \mu \frac{U}{h} \text{ (一定)}$$



上の壁を動かすのに必要な力は単位長さ当たり $\tau = \mu U/h$
速度Uで動いているので、単位時間あたりの仕事(仕事率)は
 $U\tau = \mu U^2/h$

この仕事は、流体を変形させるのに使われて熱に変わる。
上下の壁が断熱壁であれば、流体の温度が上がることになる。
(流体の運動は一定なので運動エネルギーは変化なし)

粘性流体の運動では、流体の変形によるエネルギーの損失がある。