第7回目の内容

非圧縮粘性流体の運動

- 粘性流体に働く応力
- 粘性流体の流れの支配方程式
- 粘性流体の流れの例

粘性流体に働く応力

理想流体（非粘性流体）では、流体要素に働く応力は圧力による垂直応力だけ。流体要素の変形に伴う応力はない。

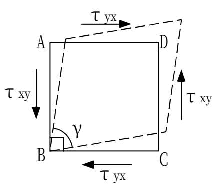
粘性流体では、圧力による垂直応力に加えて、「変形に抵抗する力」である粘性応力が働く。
粘性応力には検査面に平行な成分「剪断応力」と検査面に垂直な成分「垂直応力」がある。

応力の表記

τ_{xy}：x軸に垂直な検査面におけるy軸方向の応力

$\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy}$：剪断応力
$\tau_{xx}, \tau_{yy}, \tau_{zz}$：垂直応力

粘性剪断応力：面に平行に働く応力



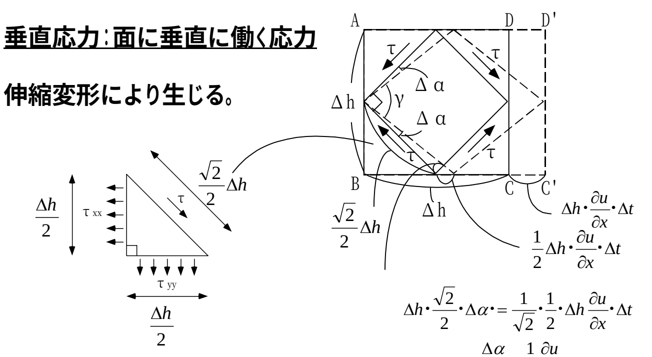
剪断変形により生じる

$$\tau_{xy} = \tau_{yx} = -\mu \cdot \frac{\partial \gamma}{\partial t} \quad \boxed{= \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}$$

μは粘性係数（流体の物性値）

垂直応力：面に垂直に働く応力

伸縮変形により生じる。



$$\Delta h \cdot \frac{\sqrt{2}}{2} \cdot \Delta\alpha \cdot = \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \cdot \Delta h \frac{\partial u}{\partial x} \cdot \Delta t$$

$$\frac{\Delta\alpha}{\Delta t} = \frac{1}{2}\frac{\partial u}{\partial x}$$

$$\frac{\partial \alpha}{\partial t} = \frac{1}{2}\frac{\partial u}{\partial x}, \frac{\partial \gamma}{\partial t} = -2\frac{\partial \alpha}{\partial t}, \frac{\partial \gamma}{\partial t} = -\frac{\partial u}{\partial x}$$

$$\tau = -\mu\frac{\partial \gamma}{\partial t} = \mu\frac{\partial u}{\partial x}$$

$$\frac{\sqrt{2}}{2}\Delta h \tau \cdot \frac{1}{\sqrt{2}} = \tau_{xx} \cdot \frac{\Delta h}{2} \qquad \tau = \tau_{xx} = \mu\frac{\partial u}{\partial x}(\text{引張})$$

$$-\frac{\sqrt{2}}{2}\Delta h \tau \cdot \frac{1}{\sqrt{2}} = \tau_{yy} \cdot \frac{\Delta h}{2} \qquad -\tau = \tau_{yy} = -\mu\frac{\partial u}{\partial x}(\text{圧縮})$$

伸びる方向に引張、他の方向に圧縮の応力となる。

$$\tau_{xx} = \underbrace{\mu\frac{\partial u}{\partial x}}_{\text{x方向の伸び：引張}} - \underbrace{\mu\frac{\partial v}{\partial y}}_{\text{y方向の伸び：圧縮}} - \underbrace{\mu\frac{\partial w}{\partial z}}_{\text{z方向の伸び：圧縮}}$$

$$= 2\mu\frac{\partial u}{\partial x} - \mu\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}_{\text{非圧縮性流体では0}}$$

$$\left.\begin{aligned}\tau_{xx} &= 2\mu\frac{\partial u}{\partial x}\\ \tau_{yy} &= 2\mu\frac{\partial v}{\partial y}\\ \tau_{zz} &= 2\mu\frac{\partial w}{\partial z}\end{aligned}\right\}\text{粘性による垂直応力}$$

応力テンソル

$$\begin{pmatrix}\tau_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \tau_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \tau_{zz}\end{pmatrix} = \mu\begin{pmatrix}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\\ \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} & \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}+\frac{\partial w}{\partial z}\end{pmatrix} = \mathbf{T}$$

$$(x, y, z) = (x_1, x_2, x_3) = \mathbf{x}$$

$$(u, v, w) = (u_1, u_2, u_3) = \mathbf{u}$$

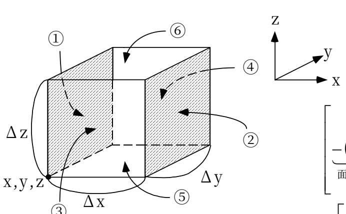
と書くと

$$\tau_{x_i x_j} = \tau_{ij} = \tau_{ji} = \mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

応力テンソル$\mathbf{T}$と検査面Aに働く応力$\mathbf{T}_A$の関係は、

$\mathbf{T}\cdot\mathbf{n}_A = \mathbf{T}_A$　($\mathbf{n}_A$は面Aの単位法線ベクトル)

流体要素に働く力＝密度・加速度の関係から支配方程式を導く



x方向に働く力は

$$\left[\underbrace{-(-p+\tau_{xx})}_{\text{面①の垂直応力}} + \underbrace{\left[-\left(p+\frac{\partial p}{\partial x}\cdot\Delta x\right)+\tau_{xx}+\frac{\partial \tau_{xx}}{\partial x}\cdot\Delta x\right]}_{\text{面②の垂直応力}}\right]\Delta y\Delta z$$

$$+\left[\underbrace{-\tau_{yx}}_{\text{面③のせん断応力}} + \underbrace{\left[\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}\cdot\Delta y\right]}_{\text{面④のせん断応力}}\right]\Delta x\Delta z$$

$$+\left[\underbrace{-\tau_{zx}}_{\text{面⑤のせん断応力}} + \underbrace{\left[\tau_{zx}+\frac{\partial \tau_{zx}}{\partial z}\cdot\Delta z\right]}_{\text{面⑥のせん断応力}}\right]\Delta x\Delta y$$

応力の勾配が単位体積に働く力となる。 →

$$= \left(-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)\Delta x\Delta y\Delta z$$

同様にy方向に作用する力　$\left(-\frac{\partial p}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)\Delta x\Delta y\Delta z$

z方向に作用する力　$\left(-\frac{\partial p}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)\Delta x\Delta y\Delta z$

流体単位体積に作用する力

$$\text{x方向}: -\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z} = -\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)+\frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)$$

$$= -\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x}+\frac{\partial^2 u}{\partial y}+\frac{\partial^2 u}{\partial z}\right)$$

$$\text{y方向}: -\frac{\partial p}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z} = -\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right)$$

$$= -\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x}+\frac{\partial^2 v}{\partial y}+\frac{\partial^2 v}{\partial z}\right)$$

$$\text{z方向}: -\frac{\partial p}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z} = -\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)+\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right)+\frac{\partial}{\partial z}\left(2\mu\frac{\partial w}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x}+\frac{\partial^2 w}{\partial y}+\frac{\partial^2 w}{\partial z}\right)$$

非圧縮　オイラー(Euler)方程式　(非粘性)

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+f_x$$
$$\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+f_y$$
$$\rho\frac{Dw}{Dt}=-\frac{\partial p}{\partial z}+f_z$$

(f_x, f_y, f_z)は単位質量あたりの外力

非圧縮　ナビエ･ストークス (Navier-Stokes) 方程式　(粘性)

$$\rho\frac{Du}{Dt}=\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+f_x$$
$$\rho\frac{Dv}{Dt}=\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+f_y$$
$$\rho\frac{Dw}{Dt}=\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+f_z$$

対流項　圧力項　粘性項　外力項

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

テンソルで書くと、

$$\rho\frac{D\mathbf{u}}{Dt}=\nabla\cdot(\boldsymbol{\tau}-p\mathbf{I})+\mathbf{f}$$ （密度・加速度＝応力の発散＋外力）

$$\nabla\cdot\mathbf{u}=0$$

$$\left[\frac{D}{Dt}=\frac{\partial}{\partial t}+\mathbf{u}\nabla,\quad \boldsymbol{\tau}=\mu\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right),\quad -p\mathbf{I}=\begin{pmatrix}-p&0&0\\0&-p&0\\0&0&-p\end{pmatrix},\quad \mathbf{f}\text{は外力ベクトル}\right]$$

連続の式を使うと、

$$\rho\frac{D\mathbf{u}}{Dt}=-\nabla p+\mu\nabla^2\mathbf{u}+\mathbf{f}$$

連続の式は粘性流体でも非粘性流体でも同じ。

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

2次元的な流れでは、一様な方向にz軸を取ると、

$w=0,\quad \dfrac{\partial}{\partial z}=0$**より**

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+f_x$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+f_y$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

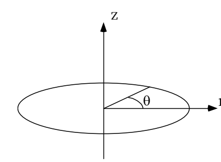
また、円筒座標系では、

非圧縮　連続の式

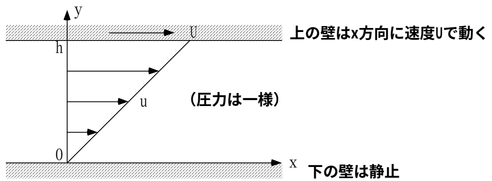
$$\frac{\partial(v_r\cdot r)}{r\partial r}+\frac{\partial v_\theta}{r\partial\theta}+\frac{\partial v_z}{\partial z}=0$$

非圧縮　$N.S.$**式**

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+v_\theta\frac{\partial v_r}{r\partial\theta}+v_z\frac{\partial v_r}{\partial z}-\frac{{v_\theta}^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)+f_r$$
$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+v_\theta\frac{\partial v_\theta}{r\partial\theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_\theta v_r}{r}=-\frac{1}{\rho}\frac{\partial p}{r\partial\theta}+\nu\left(\nabla^2 v_\theta-\frac{v_\theta}{r^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right)+f_\theta$$
$$\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+v_\theta\frac{\partial v_z}{r\partial\theta}+v_z\frac{\partial v_z}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 v_z+f_z$$
$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$$

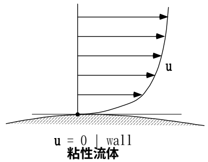


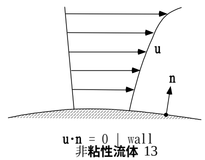
粘性流体の運動　(1) クエット流れ



粘性流体は壁面において「すべり」がない。
(壁面では流体粒子は壁と一緒に動く)

境界条件は
y = h で u = U, v = 0
y = 0 で u = 0, v = 0





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x方向の$N.S$式は、

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\quad f_x$$

外力なし $\rightarrow f_x=0$

x方向への変化なし $\rightarrow \frac{\partial}{\partial x}=0, \frac{\partial^2}{\partial x^2}=0$

時間変化なし (定常) $\rightarrow \frac{\partial}{\partial t}=0$

y方向への流れなし $\rightarrow v=0$

よって、

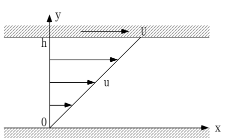
$$\mu\frac{\partial^2 u}{\partial y^2}=0$$

$$u=C_1 y+C_2$$

境界条件　$u=0\,|_{y=0}, u=U\,|_{y=h}$　より、

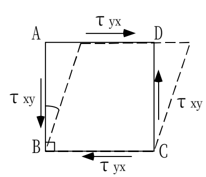
$$\underline{\underline{u=\frac{U}{h}y}}$$　直線速度分布

別の考え方 (N.S式を使わないで解くと)



x軸方向に一様な流れで、v=0であれば流体要素の伸縮はないので
(正確にはx軸方向及びy軸方向に伸縮はない) 粘性応力は、

$$\tau_{yx}=\tau_{xy}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=\mu\frac{\partial u}{\partial y}$$



右の流体要素に働く力の釣り合いより、

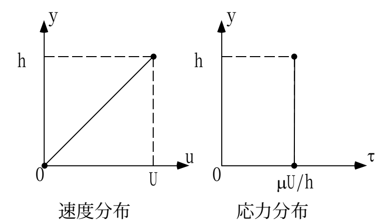
$$\frac{\partial \tau_{yx}}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)=0$$

$$\therefore \frac{\partial^2 u}{\partial y^2}=0$$

$$\tau_{yx}=\tau=\mu\frac{\partial u}{\partial y}$$

$u=\frac{U}{h}y$なので

$\tau=\mu\frac{U}{h}$ (一定)



上の壁を動かすのに必要な力は単位長さ当たりτ=μU/h
速度Uで動いているので、単位時間あたりの仕事(仕事率) は
Uτ = μU²/h
この仕事は、流体を変形させるのに使われて熱に変わる。
上下の壁が断熱壁であれば、流体の温度が上がることになる。
(流体の運動は一定なので運動エネルギーは変化なし)

<u>粘性流体の運動では、流体の変形によるエネルギーの損失がある。</u>